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# SINGULAR PERTURBATIONS FOR CONSTRAINT SYSTEMS

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SINGULAR PERTURBATIONS FOR CONSTRAINT SYSTEMS\*

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ABSTRACT

We will show a singular perturbation theorem for constraint systems, which is a generalized version of the equation;  $\dot{x} = f(x, y)$ ,  $\varepsilon \dot{y} = g(x, y)$ . At the first, we study the general properties  $G_0 \sim G_3$  of constraint systems. After this we show the properties of solutions and singular perturbation theorem for constraint system satisfying  $G_0 \sim G_3$ .

1. INTRODUCTION

The system which we want to study here was suggested by the equations of the form

$$\begin{aligned}\dot{x} &= f(x, y) \\ 0 &= g(x, y),\end{aligned}\tag{1.1}_0$$

$x \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^n$ . Many types of solutions of  $(1.1)_0$  have been studied by considering  $(1.1)_0$  as limit of

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$$\dot{x} = f(x, y)$$

(1.1)<sub>ε</sub>

$$\epsilon \dot{y} = g(x, y)$$

for  $\epsilon \rightarrow 0$ . For the studies of this type with  $m=n=1$ , there are works of B. van der Pol [14], J. LaSalle [11], A.A. Andronov and et al. [1], and others. For the case of  $m=2$  and  $n=1$ , there are works of E.C. Zeeman [15], E. Benoit [2],[3], and others. For general  $m$  and  $n$ , there are the works of L.S. Pontryagin [12], F. Takens [13], and N. Fenichel [5].

For the global version of the equation (1.1)<sub>ε</sub>, we consider a vector field  $\tilde{Z}_\epsilon/\epsilon$ , where  $\{\tilde{Z}_\epsilon\}$ ,  $\epsilon \in [0, \epsilon_0)$ , is a family of vector field on a manifold  $M$ . The limit of  $\tilde{Z}_\epsilon/\epsilon$  for  $\epsilon \rightarrow 0$  exists only on the set  $\Sigma$  of points where  $\tilde{Z}_\epsilon = 0$ , (in the case of (1.1)<sub>ε</sub>,  $\Sigma$  is the set of points where  $g(x, y) = 0$ ). But, generically in the sense of perturbations of  $\tilde{Z}$ ,  $\Sigma$  is a discrete set. To avoid this, we assume that  $\tilde{Z}_0$  is tangent to the leaves of a codimension  $m$  foliation  $\mathcal{F}$  on  $M$ .  $F$  can be considered as a generalization of the product structure  $\mathbb{R}^m \times \mathbb{R}^n$ . The vector field tangent to  $F$  is a generalization of the equation  $\dot{y} = g(x, y)$  in (1.1)<sub>ε</sub>.

A constraint system is defined as the pair  $\{\{\tilde{Z}_\epsilon\}, F\}$  as above (Definition 5.1). After the definition of the solution for a constraint system (Definition 5.4) we will define an admissible solution, which is a solution having useful properties (Definition 5.5). These definitions are motivated by F. Takens' definitions of constrained equations and the solutions [13]. Takens considered a fibre bundle structure, whereas we take a foliation. He considered a kind of

function  $M \rightarrow \mathbb{R}$  which played similar role as our vector field  $\tilde{Z}_0$  tangent to  $F$ .

Our main goal is Theorem E, Theorem F, and Theorem G. But these theorems are proved for systems having some generic properties. In section 4 we show generic properties G0, G1, and G2. G0 assures that the set of equilibrium points  $\Sigma$  of  $\tilde{Z}_0$  is a manifold. G1 is a regularity condition of the derivative of  $\tilde{Z}_0$  on  $\Sigma$ . G2 assures that  $\Sigma$  has a stratification  $S$ , which is stratified by the number of zero-eigenvalues and the number of pure imaginary eigenvalues of the derivative of  $\tilde{Z}_0|_{L_p}$  at  $p \in \Sigma$ . Here  $L_p$  is a plaque of  $F$  containing  $p$ . Theorem A in section 4 asserts that G0, G1, and G2 are generic properties. We set another property G3 in section 4, which assures that the manifold  $\Sigma$  is in general position in the foliation  $F$  with respect to Thom-Boardman singularities. Theorem B in section 4 implies that the set of  $\{\tilde{Z}_\epsilon\}$  having property G3 is dense in the space of families of vector field on  $M$  which is a subspace of  $\mathcal{X}^r(M \times [0, \epsilon_0))$ .

Saddle-node bifurcation and Hopf bifurcation are well known as typical codimension one bifurcations of equilibria. Theorem C in section 4 shows where these bifurcations of  $\tilde{Z}_0|_{L_p}$  appear for  $p \in \Sigma$ . Theorem C expresses the place in the language of the stratification  $S$  and Thom-Boardman's stratification. Theorem D determines the qualitative structure of  $\tilde{Z}_0$  near the point  $p$  where saddle-node bifurcation occurs. Theorem A, ..., Theorem D in section 4 are proved in [8].

Theorem E and Theorem F in section 5 shows the properties of admissible solutions. Theorem G is the singular perturbation theorem

for admissible solutions. This is an extension, in some senses, of L.S. Pontryagin [12] and N. Fenichel [5]; see Remark 5.9. Theorem E, Theorem F, and Theorem G are proved in [10].

In the case that  $\Sigma$  has codimension one (i.e.  $m=1$ ), it is trivial to see that the jumping path (trace of Definition 5.7) leaving a fold point is unique. When  $m>1$ , the uniqueness and other properties of the jumping path are obtained by Theorem D as the properties of the stable sets.

There is an example of constraint system in the theory of LC-network perturbation (G. Ikegami [7],[9]). In this theory, there is a foliation  $F$  (not a trivial product structure  $\mathbb{R}^m \times \mathbb{R}^n$ ) and a one parameter family of vector spaces,  $\tilde{Z}_\varepsilon = \varepsilon X + Y$  such that  $Y$  is tangent to  $F$ .

## 2. PRELIMINARIES

Let  $M$  be a smooth ( $C^\infty$ ) manifold with dimension  $m+n$ , and be a smooth foliation on  $M$  with codimension  $m$ .  $F$  is a disjoint decomposition of  $M$  into  $n$  dimensional injectively immersed connected smooth submanifolds (leaves) such that  $M$  is covered by  $C^\infty$  charts

$$\alpha_1 \times \alpha_2 : U \rightarrow D^m \times D^n \quad (2.1)$$

and  $(\alpha_1 \times \alpha_2)^{-1}(\{x\} \times D^n)$  is included in the leaf through  $(\alpha_1 \times \alpha_2)^{-1}(x, y)$ ,  $y \in D^n$ , where  $D^m$  and  $D^n$  are the open disks in  $\mathbb{R}^m$  and  $\mathbb{R}^n$ , resp. We denote

$$(\alpha_1 \times \alpha_2)^{-1}(\{x\} \times D^n) = L_{(x,y)},$$

and call it the plaque containing the point  $(x, y)$ .

Let  $\tau : TF \rightarrow M$  be the subbundle of the tangent bundle  $TM \rightarrow M$  such that the fibre  $\tau^{-1}(p)$  is an  $n$ -dimensional vector space which is tangent to the leaf of  $F$  through  $p \in M$ . Let  $Y : M \rightarrow TF$  be a  $C^r$  section of the vector bundle  $\tau$ .  $Y$  is also a  $C^r$ -section of the tangent bundle  $TM \rightarrow M$ . We call such a section a  $C^r$  vector field on  $M$  tangent to the foliation  $F$ . Denote by  $\mathcal{V}^r(F)$  the space of all  $C^r$  vector field tangent to  $F$  with the Whitney  $C^r$  topology.

We write  $\Sigma_Y$  for the subset of equilibrium points of a vector field  $Y \in \mathcal{V}^r(F)$ . A point  $p \in \Sigma_Y$  is called a regular point, if the derivative  $dY$  at  $p$  has the maximal rank  $n$ .  $p \in \Sigma_Y$  is called a normally regular point, if  $d(Y|_{L_p})(p)$  is nondegenerate, where  $L_p$  is the plaque of  $F$  at  $p$ . We denote by  $\Sigma_r$  the set of normally regular points of  $\Sigma_Y$ . A point  $p \in \Sigma_Y$  is called a normally hyperbolic point (resp. normally stable point), if  $p$  is a hyperbolic equilibrium point (resp. stable equilibrium point) of  $Y|_{L_p}$ . We write  $\Sigma_h$  (resp.  $\Sigma_s$ ) the set of normally hyperbolic (resp. stable) points. We have

$$\Sigma_s \subset \Sigma_h \subset \Sigma_r \subset \Sigma_Y.$$

Let  $\partial\Sigma_h$  be the set of all frontiers of  $\Sigma_h$ ;  $\partial\Sigma_h = \overline{\Sigma_h} - \Sigma_h$ .

A stratification  $S$  of a topological space  $N$  is a partition of  $N$  into subsets, which will be called the strata of  $S$ , such that the following conditions are satisfied:

(a) Each stratum  $S$  is locally closed, i.e. each point  $s \in S$  has a neighborhood  $U$  such that  $U \cap S$  is closed in  $U$ .

(b)  $S$  is locally finite, i.e. each point has a neighborhood meeting only finitely many strata.

(c) If  $S_1$  and  $S_2$  are strata and  $\overline{S_1} \cap S_2 \neq \emptyset$ , then  $S_2 \subset \overline{S_1}$ .

The relation  $S_2 < S_1$  defined by  $S_2 \subset \overline{S_1}$ ,  $S_2 \not\subset S_1$ , is an order on  $S$ . It is transitive and cannot have both  $S_2 < S_1$  and  $S_1 < S_2$ .

Let  $\tilde{N}$  be a  $C^1$  manifold, let  $N \subset \tilde{N}$ , and let  $S$  be a stratification of  $N$ . We will say that  $S$  is a Whitney stratification if each stratum is a  $C^1$  submanifold, and if  $S_1, S_2$  are two strata with  $S_2 < S_1$ , then for all  $x \in S_2$  the triple  $(S_1, S_2, x)$  satisfies the following Whitney's regularity condition.

Condition: For any sequences  $\{x_i\}$  of points in  $S_2$  and  $\{y_i\}$  of points in  $S_1$ , such that  $x_i \rightarrow x$ ,  $y_i \rightarrow x$ ,  $x_i \neq y_i$ , segment  $\overline{x_i y_i}$  converges (in projective space), and the tangent space  $T_{x_i} S_1^1$  converges (in Grassmanian of  $(\dim S_1)$ -plane in  $\mathbb{R}^n$ ,  $n = \dim N$ ), we have  $\ell \subset T_\infty$ , where  $\ell = \lim \overline{x_i y_i}$  and  $T_\infty = \lim T_{x_i} S_1^1$ .

Let  $S^i$  denote the substratification of a stratification  $S$  such that  $S^i$  consists of all strata of dimension  $\leq i$  of  $S$ . We call  $S^i$  the  $i$ -skeleton.

### 3. THOM-BOARDMAN SINGULARITIES MODULO FOLIATION

Suppose  $L, N$  are smooth manifold and  $f, g: L \rightarrow N$  are  $C^k$  maps with  $f(p) = g(p) = q$ .  $f$  has first order contact with  $g$  at  $p$  if  $(df)_p = (dg)_p$  as mapping  $T_p L \rightarrow T_q N$  of tangent spaces.  $f$  has  $k$ th order contact with  $g$  at  $p$  if  $(df): T_p L \rightarrow T_q N$  has  $(k-1)$ st order contact with  $(dg)$  at every point in  $T_p L$ .

Let  $M$  be a smooth manifold of dimension  $m+n$ , and let  $F$  be

a smooth foliation on  $M$  with codimension  $m$ . Let  $L$  be a smooth manifold without boundary.

Definition 3.1. Suppose  $f, g: L \rightarrow M$  are  $C^k$  maps with  $f(p) = g(p) = q$ .  $f$  is said to have  $k$ th order contact modulo  $F$  with  $g$  at  $p$  if, for some (and hence for any) chart  $(U, \alpha_1 \times \alpha_2)$  of  $F$  with  $q \in U$  given by (2.1),  $\alpha_1 \circ f: L \rightarrow D^m$  has  $k$ th order contact with  $\alpha_1 \circ g$  at  $p$ . This is written as  $f \sim_k g \text{ mod } F$  at  $p$ . Let  $J^k(L, M; F)_{p,q}$ ,  $k \geq 1$ , denote the set of equivalence classes under " $\sim_k \text{ mod } F$  at  $p$ " of mappings  $f: L \rightarrow M$  where  $f(p) = q$ . Let  $J^0(L, M; F)_{p,q} = \{(p, q)\}$ . Let  $J^k(L, M; F) = \bigcup_{(p,q) \in L \times M} J^k(L, M; F)_{p,q}$  (disjoint union). We call  $J^k(L, M; F)$  a jet space modulo  $F$ . An element  $\sigma$  in  $J^k(L, M; F)$  is called a  $k$ -jet modulo  $F$  of mapping from  $L$  to  $M$ .

For a  $C^k$  mapping  $f: L \rightarrow M$ , a jet extension

$$j^k f: L \rightarrow J^k(L, M; F)$$

is defined by stipulating that  $j^k f(x)$  is the  $k$ -jet mod  $F$  of  $f$  at  $x \in L$ .

Our jet spaces modulo foliations follow the J.M. Boardman's theory [4]. Hence, we have the following.

Proposition 3.2. For each sequence  $I = (i_1, i_2, \dots, i_k)$  of integers, the submanifold (not necessarily closed)  $\tilde{\Sigma}^I$  of the jet space modulo foliation  $J^k(L, M; F)$  is defined.  $\tilde{\Sigma}^I$  is empty unless  $I$  satisfies

$$i_1 \geq i_2 \geq \dots \geq i_{k-1} \geq i_k \geq 0,$$



$$\ell \geq i_1 \geq \ell - m,$$

$$\text{if } i_1 = \ell - m, \text{ then } i_1 = i_2 = \dots = i_k.$$

Proposition 3.3. If  $f : L \rightarrow M$  is a map whose jet section modulo  $F$ ,  $j^k f : L \rightarrow J(L, M; F)$  is transverse to  $\tilde{\Sigma}^I$ , then  $\tilde{\Sigma}^I(f) \equiv (j^k f)^{-1}(\tilde{\Sigma}^I)$  is a submanifold of  $L$ . If  $I, i$  denotes the extended sequence  $(i_1, i_2, \dots, i_k, i)$ , we have  $\tilde{\Sigma}^{I, i}(f) = \tilde{\Sigma}^i(f|_{\tilde{\Sigma}^I(f)})$ . Also, when  $I = \phi$ ,  $\tilde{\Sigma}^i(f) = \{p \in L : \dim \text{Ker } j^1 f(p) = i\}$ .

Proposition 3.4. Any map  $f : L \rightarrow M$  of class  $C^{r+1}$  may be  $C^{r+1}$  approximated in the  $C^{r+1}$  sense by a map  $g : L \rightarrow M$  whose  $r$ -jet extension  $j^r g : L \rightarrow J^r(L, M; F)$  is transverse to all submanifolds  $\tilde{\Sigma}^{i_1, \dots, i_s}$ ,  $1 \leq s \leq r$ .

We call  $\tilde{\Sigma}^I$  the Thom-Boardman submanifold of  $J^r(L, M; F)$  associated with Thom-Boardman symbol  $I$ .

These definitions and propositions in this section are described in [8].

#### 4. GENERIC PROPERTIES OF VECTOR FIELDS TANGENT TO $F$ .

In this section we introduce some theorems obtained by Ikegami [8].

Definition 4.1. Let  $\dim M = m + n$  and  $\text{codim } F = m$ . The following are the properties of the vector field  $Y \in \mathcal{V}^r(M, F)$ .

G0: The set  $\Sigma_Y$  of all equilibrium points of  $Y$  is, if nonempty, an  $m$  dimensional  $C^r$  manifold.

G1: Every point of  $\Sigma_Y$  is regular.

G2:  $Y$  has the property  $G_0$  and there is a Whitney stratification  $S$  on  $\Sigma_Y$  having the following properties:

(i) If the differential  $d(Y|_{L_p})(p)$  at  $p$  has  $\ell$  eigenvalues of zero and  $2(k-\ell)$  non-zero pure imaginary eigenvalues

$$0, \dots, 0, ib_1, -ib_1, \dots, ib_{k-\ell}, -ib_{k-\ell},$$

then  $p$  is contained in the  $(m-k)$  skeleton  $S^{m-k}$ .

(ii) The union of all  $(m-1)$  dimensional strata  $\cup S^{m-1}$  is a dense subset of  $\partial \Sigma_h$ .

(iii)  $\cup S^{m-1}$  is divided into two parts,  $(\partial \Sigma_h)_0$  and  $(\partial \Sigma_h)_{img}$ , of unions of strata such that

$$p \in (\partial \Sigma_h)_0 \implies 0 \text{ is an eigenvalue of } d(Y|_{L_p})(p),$$

$p \in (\partial \Sigma_h)_{img} \implies$  the eigenvalues of  $d(Y|_{L_p})(p)$  include a pair of non-zero pure imaginary numbers.

G3:  $Y$  has the property  $G_0$ , and for  $k=1, 2$ , the  $k$ -jet extension  $j^k \iota: \Sigma_Y \rightarrow J^k(\Sigma_Y, M; F)$  of the inclusion map  $\iota: \Sigma_Y \rightarrow M$  is transverse to  $\tilde{\Sigma}^I$  for all Thom-Boardman submanifold  $\tilde{\Sigma}^I$  of length  $k$  symbol  $I$ .

Let  $\mathcal{Y}_k^r$  denote the set of  $Y \in \mathcal{Y}^r(M, F)$  satisfying the property  $G_k$ ,  $k=0, 1, 2, 3$ .

Theorem A. For  $k=0, 1, 2$ , the set  $\mathcal{Y}_k^r$  is open dense in  $\mathcal{Y}^r(M; F)$ , if  $k+1 \leq r < \infty$ .

Theorem B.  $\mathcal{Y}_3^r$  is dense in  $\mathcal{Y}^r(M; F)$  for  $3 \leq r < \infty$ .

Let  $\iota: \Sigma_Y \rightarrow M$  be the inclusion map. Let  $\tilde{\Sigma}^I \subset J^k(\Sigma_Y, M; F)$  be the Thom-Boardman manifold for Thom-Boardman symbol I. Denote  $\tilde{\Sigma}^I(Y) \equiv (j^k \iota)^{-1}(\tilde{\Sigma}^I)$ .

Let  $\tau: TF \rightarrow M$  be the vector bundle of vectors tangent to  $F$ . Let  $(\alpha, \alpha_1 \times \alpha_2, U)$  be a vector bundle chart of  $\tau$ . Let  $J^1(\tau)$  be the 1-jet space of germs of partial sections of  $\tau$ . Define  $\tilde{\Sigma}_\tau^1$  to be the set of 1-jet  $\sigma \in J^1(\tau)$  such that, if  $Y$  represents  $\sigma$  at  $p \in M$ , then  $Y(p) = 0$  and the rank of  $d(Y|_{L_p})(p) = n - i$ . Denote  $\tilde{\Sigma}_\tau^1(Y) \equiv (j^1 Y)^{-1}(\tilde{\Sigma}_\tau^1)$ .

The following can be easily proved [8].

Proposition 4.2. Let  $Y \in \mathcal{Y}^r(M; F)$ ,  $r \geq 2$ . Then we have the following.

- (i)  $\tilde{\Sigma}_\tau^1(Y) = \tilde{\Sigma}^1(Y)$ , if  $Y$  satisfies G0 and G1.
- (ii) If  $Y$  satisfies G3, then each point  $p \in \tilde{\Sigma}^{1,0}(Y)$  is a fold point; i.e. there exist coordinates of class  $C^{r-1}$ ,  $x_1, \dots, x_m$  centered at  $p$  in  $\Sigma_Y$  and  $y_1, \dots, y_m, z_1, \dots, z_n$  centered at  $p$  in  $M$ , such that (a)  $z_1, \dots, z_n$  are the coordinates of the plaque  $L_p$  of  $F$ , (b) the inclusion map  $\Sigma_Y \rightarrow M$  is given by

$$y_1 = x_1, \dots, y_{m-1} = x_{m-1}, y_m = x_m^2;$$

$$z_1 = x_m, \quad z_2 = \dots = z_n = 0.$$

This proposition is useful in the proofs of Theorem C and Theorem D below in this section.

Next, we study the bifurcations of  $Y$  at  $\Sigma_h$ . Suppose that

$\dim M = m+n$ ,  $\text{codim } F = m$ , and  $Y$  is of class  $C^r$ ,  $r \geq 3$ . Let  $p$  be a point in  $\partial \Sigma_h$ . Assume that there is a neighborhood  $N$  of  $p$  in  $\partial \Sigma_h$  such that  $N$  is an  $(m-1)$  dimensional manifold. Let  $\alpha_1 \times \alpha_2 : U \rightarrow D^m \times D^n$  be a chart of  $F$  such that  $(\alpha_1 \times \alpha_2)(p) = (0,0)$ , (see (2.1)). Let  $I$  be a segment in  $D^m$  parametrized by  $\mu$  such that  $\mu=0$  indicates the origin of  $D^m$ .

Assumption:  $L \equiv (\alpha_1 \times \alpha_2)^{-1}(I \times D^n)$  is transverse to both  $\Sigma_Y$  and  $N$  in  $M$ .

Definition 4.3. Under the above assumption we say that  $Y$  has saddle-node bifurcation at  $p \in \partial \Sigma_h$ , if there is an segment  $I$  as above satisfying the following: The smooth curve  $L \cap \Sigma_Y$  is tangent to  $L_0$  at  $p$ ,  $\Sigma_Y \cap L_\mu = \emptyset$  if  $\mu < 0$ , and  $\Sigma_Y \cap L_\mu$  consists of two points,  $p_\mu^s$  and  $p_\mu^u$  if  $\mu > 0$ . Furthermore,  $Y$  is hyperbolic at  $p_\mu^s$  and  $p_\mu^u$ . The dimensions of the stable manifolds at  $p_\mu^s$  and  $p_\mu^u$  are  $k$  and  $k-1$ , respectively,  $1 \leq k \leq m$ . See Figure 1.

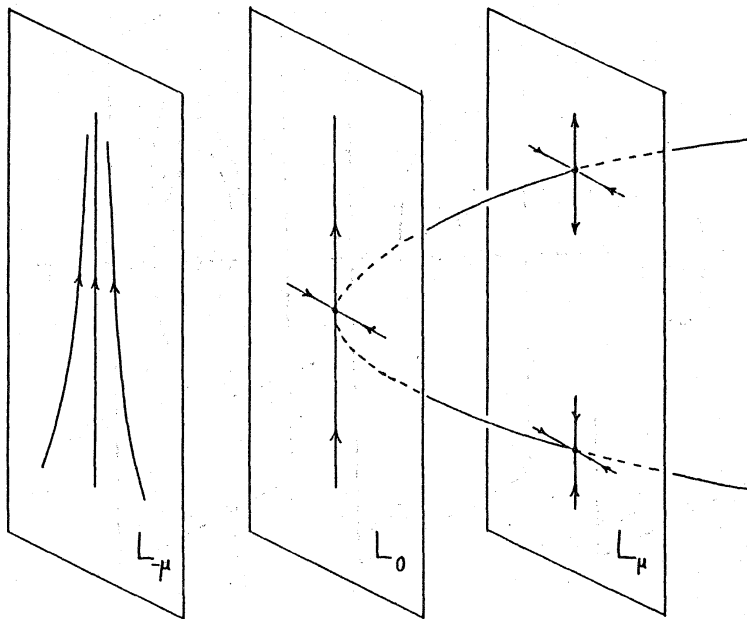


Figure 1

Definition 4.4. Under the above assumption we say that  $Y$  has Hopf bifurcation at  $p \in \partial \Sigma_h$ , if the following hold for every segment  $I \subset D^m$  as above: There is a unique 3-dimensional center manifold  $C$  (see Guckenheimer-Holmes [6, p.127]) containing  $L \cap \Sigma_Y = (\cup_{\mu} L_{\mu}) \cap \Sigma_Y$  and a system of coordinates  $(x, y, \mu)$  on  $C$ , with  $(x, y, \mu) \in L_{\mu}$ , for which the Taylor expansion of degree 3 of  $Y$  on  $C$  is given by

$$\begin{cases} \dot{x} = (d\mu + a(x^2 + y^2))x - (\omega + c\mu + b(x^2 + y^2))y \\ \dot{y} = (\omega + c\mu + b(x^2 + y^2))x + (d\mu + a(x^2 + y^2))y, \end{cases}$$

which is expressed in polar coordinates as

$$\begin{cases} \dot{r} = (d\mu + ar^2)r \\ \dot{\theta} = (\omega + c\mu + br^2). \end{cases}$$

See Figure 2. Consequently, if  $a \neq 0$ , there is a surface of periodic

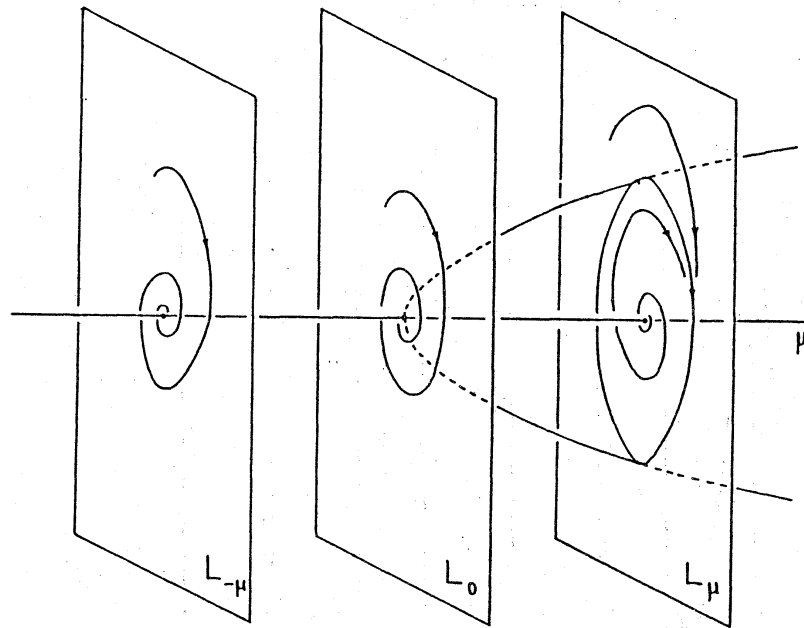


Figure 2

solutions in  $C$  which has quadratic tangency with the eigenspace of  $\lambda(0)$ ,  $\bar{\lambda}(0)$  agreeing to second order with the paraboloid  $\mu = -(a/d)(x^2 + y^2)$ . If  $a < 0$ , these solutions are stable limit cycles, while if  $a > 0$ , there are repelling. (See [6, Theorem 3.4.2].)

Saddle-node bifurcation and Hopf bifurcation are well known as typical codimension one bifurcations of equilibria (e.g. [6]). We want to see how these bifurcations arise in our global situation with respect to the stratifications which we defined. The stratification  $S$  in  $G_2$  is defined by only the first derivatives of  $Y$ . But, saddle-node bifurcation does not occur under the condition only of the first derivatives. As another condition we take the second derivatives modulo  $F$  of the inclusion map of the set of equilibrium points  $\Sigma_Y$ ; while J. Guckenheimer and P. Holmes [6, Theorem 3.4.1] take the assumption for the second derivative of  $Y$ . For this purpose, we use the stratification of Thom-Boardman. In the study of constraint systems, it is natural to consider Thom-Boardman singularities (see [13] and [15]).

Let  $S^k$  be the  $k$ -skeleton of  $S$ . Let  $\tilde{S}^k$  be the  $k$ -skeleton of the stratification determined by  $\tilde{\Sigma}^i(Y) = (j^1_1)^{-1}(\tilde{\Sigma}^i)$ ,  $i = 0, 1, \dots, m$ . We have  $\tilde{S}^k = \tilde{\Sigma}^{m-k}(Y) \cup \tilde{\Sigma}^{m-k+1}(Y) \cup \dots \cup \tilde{\Sigma}^m(Y)$ . Under  $G_1$ ,  $S^k \supset \tilde{S}^k$  and  $S^{m-1} = \partial \Sigma_h$  hold by Proposition 4.2(i) and the definition of  $S$ . Moreover, we have that a  $(m-1)$  dimensional stratum of  $S$  is included in a  $(m-1)$  dimensional stratum of  $\tilde{S}$ . For the sets defined in  $G_2$ , we observe

$$(\partial \Sigma_h)_0 \subset \tilde{S}^{m-1} \quad \text{and} \quad (\partial \Sigma_h)_{\text{img}} \cap \tilde{S}^{m-1} = \phi.$$

Denote by  $(\partial\Sigma_h)_f$  the set of fold points in  $\partial\Sigma_h$ ;

$$(\partial\Sigma_h)_f \equiv (\partial\Sigma_h)_0 \cap \tilde{\Sigma}^{1,0}(Y)$$

Theorem C. Let  $Y \in \mathcal{V}^r(F)$ ,  $r \geq 3$ . Suppose that  $Y$  satisfies G1, G2, and G3. Then, there is an open dense subset  $(\partial\Sigma_h)_f \cup (\partial\Sigma_h)_{img}$  of the boundary  $\partial\Sigma_h$  of the normally hyperbolic domain  $\Sigma_h \subset \Sigma_Y$  such that  $Y$  has saddle-node bifurcation at each point of  $(\partial\Sigma_h)_f$  and has Hopf bifurcation at each point of  $(\partial\Sigma_h)_{img}$ .

Next, we study the qualitative structure of  $Y$  at fold points in the boundary of normally stable domain  $\Sigma_s$ .

Let  $X$  be a  $C^r$  vector field on an open set  $U$  in  $\mathbb{R}^n$ , let  $\phi_t$  be the flow of  $X$ , and let  $p \in U$  be an equilibrium point of  $X$ . Suppose that the eigenvalues  $\lambda_0, \dots, \lambda_{n-1}$  of  $dX(p)$  satisfy that  $\lambda_0 = 0$  and that the real parts  $\text{Re} \lambda_1, \dots, \text{Re} \lambda_{n-1} < 0$ . Let  $E^c$  and  $E^s$  be the generalized eigen spaces of  $\lambda_0$  and  $\lambda_1, \dots, \lambda_{n-1}$ , respectively. By center manifold theorem (Guckenheimer-Holmes [6, Theorem 3.2.1]), there are an invariant  $C^r$  manifold  $W^s(p)$  (called the stable manifold) tangent to  $E^s$  at  $p$  and a  $C^r$  manifold  $W^c(p)$  (called the (local center manifold)) tangent to  $E^c$  at  $p$ .  $W^c$  is locally invariant in the sense that, if  $q \in W^c$  and  $\phi_t(q) \in U$ , then  $\phi_t(q) \in W^c$ .  $W^s$  is unique, but  $W^c$  need not be.

Let  $\psi_t$  be the flow associated to a vector field on a manifold. The subsets

$$V^s(p) = \{q : \psi_t(q) \rightarrow p \text{ as } t \rightarrow \infty\}, \text{ and}$$

$$V^u(p) = \{q : \psi_t(q) \rightarrow p \text{ as } t \rightarrow -\infty\}$$

are called the stable set and the unstable set of  $p$ , respectively.

The boundary  $\partial\Sigma_s = \overline{\Sigma_s} - \Sigma_s$  of normally stable domain is included in the boundary  $\partial\Sigma_h$  of normally hyperbolic domain. Suppose  $Y$  satisfies G1, G2, and G3. Then, by Theorem C, there is an open dense subset  $(\partial\Sigma_h)_f \cup (\partial\Sigma_h)_{img}$  of  $\partial\Sigma_h$  such that  $Y$  has saddle-node bifurcation at  $(\partial\Sigma_h)_f$  and has Hopf bifurcation at  $(\partial\Sigma_h)_{img}$ . Define the sets as follow,

$$(\partial\Sigma_s)_f \equiv (\partial\Sigma_h)_f \cap (\partial\Sigma_s) \quad \text{and} \quad (\partial\Sigma_s)_{img} \equiv (\partial\Sigma_h)_{img} \cap (\partial I_s).$$

Theorem D. Suppose  $Y \in \mathcal{V}^r(M; F)$ ,  $r \geq 3$ . Let  $(\partial\Sigma_s)_f \cup (\partial\Sigma_s)_{img}$  be the open dense subset of  $\partial\Sigma_s$  defined as above. Let  $p \in (\partial\Sigma_s)_f$ .

Then, there are an open neighborhood  $U$  of  $p$  in  $M$  and a  $C^r$  embedding from the plaque,  $h_p : L_p \rightarrow \mathbb{R}^1 \times \mathbb{R}^{n-1}$  such that the following are satisfied.

(i)  $W^s(p) \cap L_p = h_p^{-1}(\{0\} \times \mathbb{R}^{n-1})$  and  $W^c(p) \cap L_p \subset h_p^{-1}(\mathbb{R}^1 \times \{0\})$ , where  $W^s(p)$  and  $W^c(p)$  are the stable and center manifold of  $Y|_{L_p}$ , respectively.

(ii)  $V^s(p) \cap L_p \subset h_p^{-1}([0, \infty) \times \mathbb{R}^{n-1})$  and  $V^u(p) \cap L_p \subset h_p^{-1}((-\infty, 0] \times \{0\}) \subset W^c(p)$ , where  $V^s(p)$  and  $V^u(p)$  are the stable and unstable sets of  $p$ , respectively. (Figure 3).

(iii) The  $C^r$  embedding  $h_p$  depends  $C^{r-1}$  continuously on  $p \in (\partial\Sigma_s)_f$ . So that, both of the sets

$$V^u = \{q \in V^u(p) : p \in (\partial\Sigma_s)_f \cap U\}$$

and  $V^u(p)$  are injectively  $C^{r-1}$  immersed submanifolds of  $M$ .



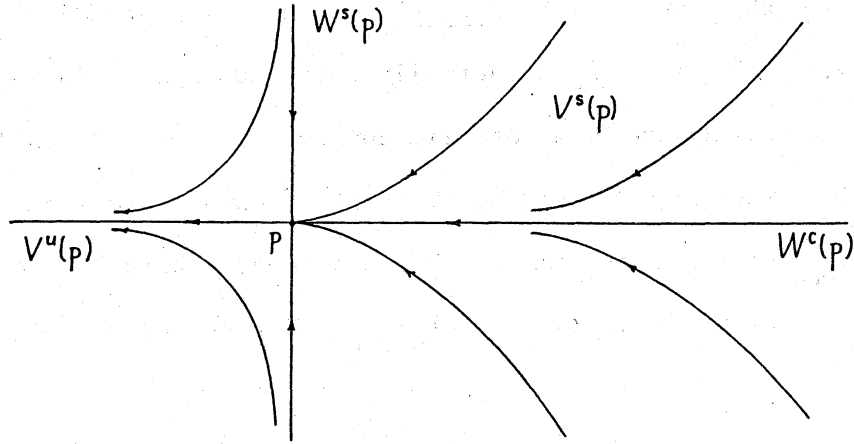


Figure 3

### 5. CONSTRAINT SYSTEMS AND SINGULAR PERTURBATIONS

Let  $M$  be a smooth manifold. Let  $\{\tilde{Z}_\varepsilon\}$ ,  $0 \leq \varepsilon < \varepsilon_0$ , be a family of vector fields on  $M$ .  $\{\tilde{Z}_\varepsilon\}$  is called a  $C^r$  family if  $\tilde{Z}_\varepsilon(p)$  is a  $C^r$  vector field on  $M \times [0, \varepsilon_0)$ . In this section, we assume  $r \geq 3$ .

Definition 5.1. A constraint system of class  $C^r$  on  $M$  is a pair  $\{\{\tilde{Z}_\varepsilon\}, F\}$  of  $C^r$  family of vector fields on  $M$ ,  $\{\tilde{Z}_\varepsilon\}$   $0 \leq \varepsilon < \varepsilon_0$  and a smooth foliation  $F$  on  $M$  such that  $\tilde{Z}_0$  ( $\varepsilon=0$ ) is tangent to (the leaves of)  $F$ . We may call the limit of  $\tilde{Z}_\varepsilon/\varepsilon$  for  $\varepsilon \rightarrow 0$  a constrained equation in different meaning from Takens [13]. This limit exists only at most on the subset of equilibrium points of  $\tilde{Z}_0$ .

Expanding  $\tilde{Z}_\varepsilon$  by  $\varepsilon$ , we have

$$\left. \begin{aligned} \tilde{Z}_\varepsilon(p) &= Y(p) + \varepsilon \cdot X(p) + o(\varepsilon) \\ Y(p) &= \tilde{Z}_0(p) \\ X(p) &= \frac{\partial}{\partial \varepsilon} \tilde{Z}_\varepsilon(p) \Big|_{\varepsilon=0} \end{aligned} \right\} \quad (5.1)$$

We set a following axiom for  $\{\{\tilde{Z}_\varepsilon\}, F\}$ .

Axiom 5.2.  $Y = \tilde{Z}_0$  satisfies G1, G2, and G3.

Remark 5.3. By Theorem A and Theorem B, the set of families satisfying Axiom is dense in the space  $Z^r$  of  $C^r$  family of vector fields  $\{\tilde{Z}_\varepsilon\}$  such that  $\tilde{Z}_0$  is tangent to  $F$ . Here,  $Z^r$  is defined usually as a subspace of the space  $\mathcal{X}^r(M \times [0, \varepsilon_0])$  of  $C^r$  vector fields on  $M \times [0, \varepsilon_0]$ .

Let  $\Sigma_r$  be the normally regular domain of the manifold  $\Sigma_Y$  of equilibrium points of  $Y = \tilde{Z}_0$ . Hereafter, we use the simple notation  $\Sigma$  for  $\Sigma_Y$ . Let

$$\begin{array}{ccc} T_{\Sigma_r} M & \xrightarrow{\pi_\Sigma} & T_{\Sigma_r} \Sigma \\ \text{Proj.} \searrow & \curvearrowright & \swarrow \text{Proj.} \\ & \Sigma_r & \end{array}$$

be the bundle map obtained by the projection

$$T_p M = T_p \Sigma_r \oplus T_p L_p \longrightarrow T_p \Sigma_r$$

for each  $p \in \Sigma_r$ , where  $L_p$  is the plaque of  $F$  containing  $p$ . For a crosssection  $X$  of the bundle  $T_{\Sigma_r} M \rightarrow \Sigma_r$ , we define a vector field  $X_\Sigma$  on  $\Sigma_r$  by

$$X_\Sigma \equiv \pi_\Sigma X \quad (5.2)$$

Definition 5.4. A curve  $\gamma : (a, b) \rightarrow \Sigma_r$  is a solution of the constrained equation  $\lim_{\varepsilon \rightarrow 0} \tilde{Z}_\varepsilon / \varepsilon$  associated with  $\{\{\tilde{Z}_\varepsilon\}, F\}$  if

(i)  $\lim_{t \rightarrow t_0} \gamma(t) = \gamma(t_0)$  and there is  $\lim_{t \rightarrow t_0} \gamma(t) \equiv \gamma^-(t_0)$  in  $\Sigma$  (not necessarily in  $\Sigma_r$ );

(ii) whenever  $\gamma^-(t_0) \neq \gamma(t_0)$ , there is an orbit  $C$  (included in a leaf of  $F$ ) of  $\tilde{Z}_0$  such that the  $\alpha$  limit set  $\alpha(C)$  and the  $\omega$  limit set  $\omega(C)$  of  $C$  satisfy

$$\alpha(C) = \gamma^-(t_0) \quad \text{and} \quad \omega(C) = \gamma(t_0);$$

(iii) if  $\gamma^-(t_0) = \gamma(t_0)$ , then  $X_\Sigma \gamma(t_0)$  is the derivative of  $\gamma$  at  $t_0$ ; if  $\gamma^-(t_0) \neq \gamma(t_0)$ , then  $X_\Sigma \gamma(t_0)$  is the right derivative of  $\gamma$  at  $t_0$ .

A curve  $\gamma : [a, b) \rightarrow \Sigma_r$  is a solution if, (i) for any  $a < a' < b$ ,  $\gamma|_{[a', b)}$  is a solution; (ii)  $X_\Sigma \gamma(a)$  is the right derivative of  $\gamma$  at  $a$ .

A curve  $\gamma : (a, b] \rightarrow \Sigma_r$  is a solution if, (i) for any  $a < b' < b$ ,  $\gamma|_{(a, b']}$  is a solution; (ii) there is  $\lim_{t \rightarrow b} \gamma(t) = \gamma^-(b)$  in  $\Sigma$ ; (iii) there is an orbit  $C$  of  $\tilde{Z}_0$  such that  $\alpha(C) = \gamma^-(b)$  and  $\omega(C) = \gamma(b)$ .

$\gamma : [a, b] \rightarrow \Sigma_r$  is a solution if  $\gamma|_{[a, c)}$  and  $\gamma|_{(c, b]}$  are solution for any  $a < c < b$ .

For a point  $p \in \Sigma_r$ , there is a solution  $\gamma : (a, b) \rightarrow \Sigma_r$  such that  $p = \gamma(c)$ ,  $a < c < b$ . But there may be many such solutions. See Figure 4 and 5.

Next, we consider solutions having many available properties.

Let  $\tilde{Z}_\epsilon = Y + \epsilon X + o(\epsilon)$ . Let  $\Sigma$  be the set of equilibrium points of  $Y$ .

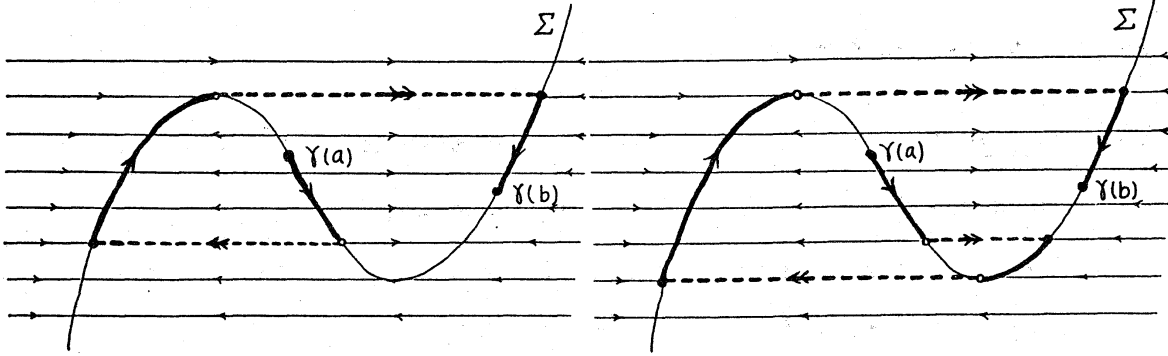


Figure 4

Figure 5

**Definition 5.5.** Let  $J$  be an interval. A solution  $\gamma: J \rightarrow \Sigma_r$  of  $\lim_{\varepsilon \rightarrow 0} \tilde{Z}_\varepsilon/\varepsilon$  is called to be admissible if

- (i) the image  $\gamma(J)$  is included in the normally stable domain  $\Sigma_s$  of  $Y$ ,
- (ii) whenever  $\gamma$  is not continuous at  $t \in J$  then  $p = \gamma^-(t)$  is contained in the fold point set  $(\partial \Sigma_s)_f$  in  $\partial \Sigma_s$ , and furthermore

$$X(p) \notin T_p \Sigma + T_p L_p \quad (5.3)$$

is satisfied.

**Remark 5.6.** (5.3) is a generic condition. In fact, since  $p \in (\partial \Sigma_h)_0 \subset \tilde{\Sigma}^1(Y)$ , the subspace  $T_p \Sigma + T_p L_p$  has codimension one in  $T_p M$ . Hence, by a perturbation of  $X$  (hence, of  $\tilde{Z}$ ), we have  $\tilde{Z}$  such that (5.3) holds for the points  $p$  in an open dense subset of  $(\partial \Sigma_s)_f$ .

Hereafter, we show some properties of admissible solutions. For a non-zero vector  $v \in T_p M$ , denote by  $L(v)$  the 1-dimensional subspace

of  $T_p M$  generated by  $v$ . The unstable set  $V^u(p)$  of  $p \in (\partial \Sigma_s)_f$  is an injectively immersed submanifold of  $[0, \infty)$  in  $M$ , and it exists uniquely for  $p$ , by Theorem D.

Theorem E. Let  $\tilde{Z}_\epsilon = Y + \epsilon X + o(\epsilon)$ . Suppose that  $X$  satisfies (5.3) at a point  $p \in (\partial \Sigma_s)_f$ . Then the following hold.

- (i) For some (and hence for any) Finsler  $\|\cdot\|$  on  $TM$  and  $q \in \Sigma_s$ , we have  $\|X_\Sigma(q)\| \rightarrow \infty$  ( $q \rightarrow p$ ).
- (ii) For  $q \in \Sigma_s$ , we have  $L(X_\Sigma(q)) \rightarrow T_p V^u(p)$ ,  $q \rightarrow p$ .

Theorem F. Let  $\phi_t(q)$  be the trajectory of  $\pi_\Sigma X$  on  $\Sigma_s$  such that  $\phi_0(q) = q$ . Suppose that

$$\lim_{t \nearrow a} \phi_t(q) = p \in \Sigma_f, \quad a > 0.$$

Then, the following hold.

- (i) For any point  $q'$  in a neighborhood  $U$  of  $q$  in  $\Sigma_s$ , there are  $p' \in (\partial \Sigma_s)_f$  and  $a' > 0$  such that

$$\lim_{t \nearrow a'} \phi_t(q') = p'.$$

- (ii) The mapping  $U \rightarrow (\partial \Sigma_s)_f$ , defined by  $q' \mapsto p'$ , is continuous.

Definition 5.7. Let  $\gamma: J \rightarrow \Sigma_s$  be a solution of  $\lim_{\epsilon \rightarrow 0} \tilde{Z}_\epsilon/\epsilon$ . For a discontinuous point  $t_i$ ,  $i=1, 2, 3, \dots$ , let  $C_i$  be the orbit of  $\tilde{Z}_0$  with  $\alpha(C_i) = \gamma^-(t_i)$  and  $\omega(C_i) = \gamma(t_i)$ . The arc

$$\Gamma(\gamma) \equiv \gamma(J) \cup C_1 \cup C_2 \cup C_3 \cup \dots$$

is called the trace of  $\gamma$ .

Let  $d$  be a Riemannian metric on  $M$ .

Theorem G. (Singular perturbation theorem). Let  $\gamma : [0, b] \rightarrow \Sigma_s$  be an admissible solution of a constrained equation  $\lim_{\varepsilon \rightarrow 0} \tilde{Z}_\varepsilon / \varepsilon$  such that  $\gamma$  has at most finitely many discontinuous points. Let  $\psi_\varepsilon : \mathbb{R} \times M \rightarrow M$  be the flow associated with the vector field  $Z_\varepsilon \equiv \tilde{Z}_\varepsilon / \varepsilon$ ,  $\varepsilon \neq 0$ .

Then, for any  $\delta > 0$  and  $\mu > 0$ , there exist  $\bar{\varepsilon} > 0$  and a neighborhood  $U$  of  $p = \gamma(0)$  in  $M$  such that, for any  $\varepsilon$  with  $0 < \varepsilon < \bar{\varepsilon}$  and any  $q \in U$  the following hold.

(i)  $\psi_\varepsilon(J, q)$  is included in the  $\delta$ -neighborhood of the trace  $\Gamma(\gamma)$ ; i.e. for any  $t \in J$

$$d(\psi_\varepsilon(t, q), \Gamma(\gamma)) < \delta.$$

(ii) If  $t \in J$  and  $|t - t_i| \geq \eta$  for every discontinuous points  $t_1, t_2, t_3, \dots \in J$  of  $\gamma$ , then we have

$$d(\psi_\varepsilon(t, q), \gamma(t)) < \delta.$$

Corollary 5.8. Admissible solution  $\gamma : [0, b] \rightarrow \Sigma_s$  with  $\gamma(0) = p$  is unique, i.e. if  $\gamma' : [0, b] \rightarrow \Sigma_s$  is another admissible solution with  $\gamma'(0) = p$ , then  $\gamma(t) = \gamma'(t)$  for any  $0 < t \leq b$ .

Remark 5.9. (i) N. Fenichel [5, Theorem 9.1] proves the singular perturbation theory for a neighborhood of a compact subset of normally hyperbolic domain  $\Sigma_h$ . We use this theory for the proof of Theorem G.

(ii) L.S. Pontryagin [12] shows the singular perturbation theorem in the neighborhood of a discontinuous point of  $\gamma$  under the condition of the derivatives of  $Y$ . This condition is slightly different to our

theorem which takes the condition of  $\Sigma \hookrightarrow M$ . In the proof [10] of Theorem G, we do not use Pontryagin's results; we give another proof using center manifold theorem.

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